Chapter 3 Source Coding

- 3.1 An Introduction to Source Coding
- 3.2 Optimal Source Codes
- 3.3 Shannon-Fano Code
- 3.4 Huffman Code

§ 3.1 An Introduction to Source Coding

- Entropy (e.g., in bits per symbol) implies the average number of bits that are required to represent a source symbol. This indicates a mapping between the source symbols and bits.
- **Source coding** can be seen as a mapping mechanism between source symbols and e.g., bits.
- For a string of symbols, how can we use less bits to represent them?

Intuition: Use short description to represent the most frequently occurred symbols; Use necessarily long description to represent the less frequently occurred symbols.

§ 3.1 An Introduction to Source Coding

Symbols: 1 2 4 4 3 1 4 4

⇩

bits: 00 01 11 11 10 00 11 11

⇩

Or can this be a shorter string of bits?

- **Expected Length :** Let x denote a source symbol and $C(x)$ denote a codeword of x. If the length of $C(x)$ is $l(x)$ (e.g., in bits) and x occurs with a probability of $p(x)$, the <u>expected length</u> $L(C)$ of source code C is:

$$L(C) = \sum_x p(x) \cdot l(x).$$

- It implies the average number of bits that are required to represent a source symbol in source coding scheme C.

§ 3.1 An Introduction to Source Coding

Let us look at the following example:

Example 3.1 Let X be a random variable with an alphabet of $\{1,\ 2,\ 3,\ 4\}$, it has a distribution of

$$P(x=1)=\frac{1}{2}, P(x=2)=\frac{1}{4}, P(x=3)=\frac{1}{8}, P(x=4)=\frac{1}{8}$$

Entropy of X is:

$$\begin{aligned} H(X) &= \sum_{x\in\{1,2,3,4\}} P(x)\log_2 P(x)^{-1} \\ &= 1.75 \text{ bits/sym.} \end{aligned}$$

§ 3.1 An Introduction to Source Coding

Source Coding 1 (C):

$$C(1) = 00,\ C(2) = 01,\ C(3) = 10,\ C(4) = 11$$

$$L(C) = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 2 + \frac{1}{8} \cdot 2 = 2 \text{ bits.}$$

On average, we use 2 bits to represent a symbol.

⇨ $L(C) > H(X)$.

Source Coding 2 (C^*):

$$C^*(1) = 0,\ C^*(2) = 10,\ C^*(3) = 110,\ C^*(4) = 111$$

$$L(C^*) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 = 1.75 \text{ bits}$$

On average, we use 1.75 bits to represent a symbol.

⇨ $L(C^*) = H(X)$.

Observation: C^* should be a better source coding scheme than C.

§ 3.1 An Introduction to Source Coding

Memoryless Source: Given a source symbol sequence $s_1, s_2, \dots, s_n$. It is memoryless if

$$P(s_j) = P(s_j \mid s_1, s_2, \dots, s_{j-1}), \forall j = 1, 2, \dots, n.$$

The source symbols are statistically independent.

Theorem 3.1 Shannon's Source Coding Theorem Given a memoryless source X whose symbols are chosen from the alphabet $\{x_1, x_2, \dots, x_U\}$ with the alphabet symbol probabilities of $P(x_1) = p_1, P(x_2) = p_2, \dots, P(x_U) = p_U$, and $\sum_{i=1}^{U} p_i = 1$. If the source is of length n, when $n \to \infty$, it can be encoded with $H(X)$ bits per symbol. The coded sequence will be of $nH(X)$ bits.

Note: $H(X) = \sum_{i=1}^{U} p_i \log_2 p_i^{-1}$ bits/sym.

§ 3.1 An Introduction to Source Coding

Important Features of Source Coding:

1. **Non-singularity:** Unambiguous representation of source symbols.
   That says if $x_i \neq x_j$, $c(x_i) \neq c(x_j)$.

X	$C(X)$
1	0
2	010
3	01
4	10

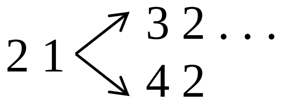
Problem: When we try to decode '010', it can be 2 or 14 or 31.

The decoding is NOT unique.

2. **Uniquely decodable:** A codeword can only be uniquely decoded into a source symbol.

X	$C(X)$
1	10
2	00
3	11
4	110

Problem: When we try to decode '001011000', we have



We will have to wait and see the end of the bit string. The decoding is NOT instantaneous.

§ 3.1 An Introduction to Source Coding

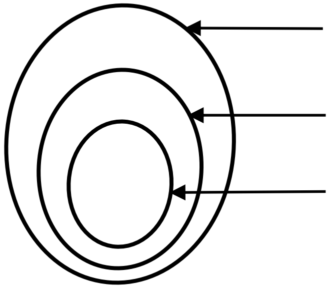
3. **Instantaneous decoding:** The decoding (demapping) happens once a codeword is read.

Instantaneous codes: For an instantaneous code, no codeword is a prefix of any other codeword.

X	$C(X)$
1	0
2	10
3	110
4	111

Observation: If you try to decode ‘111110101100111’, you would notice that the puncturing positions are determined by the instances you have reached a source codeword. The decoding is instantaneous, and the decoding output is ‘4 3 2 3 1 4 ’ .

Source Codes:



Non-singular codes

Uniquely decodable codes

Instantaneous codes

§ 3.2 Optimal Source Codes

How can we find an optimal source code?

An optimal source code :

(1) An instantaneous code (prefix code)
(2) The smallest expected length $L = \sum_i p_i l_i$

Theorem 3.2 Kraft Inequality For an instantaneous code over an alphabet of size D (e.g., $D = 2$ for binary codes), the codeword lengths $l_1, l_2, \cdots, l_U$ must satisfy

$$\sum_i D^{-l_i} \leq 1.$$

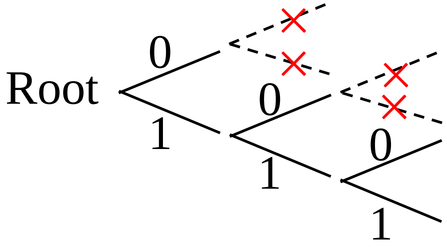
Remark: An instantaneous code $\rightleftarrows$ $\sum_i D^{-l_i} \leq 1$

Example 3.2 For the source code C^* of *Example 3.1*.

$$2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} = 1.$$

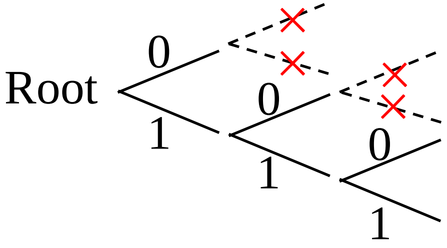
§ 3.2 Optimal Source Codes

Proof:



- Assume *D* = 2, and the above binary tree illustrates the assignment of binary source codeword. A complete solid path represents a source codeword.
- Based on property of instantaneous codes, if the first source codeword goes the ‘0’ path, the next source codeword should not go the ‘0’ path. Such a source codeword symbol assignment process repeats as the number of data symbols increases.

§ 3.2 Optimal Source Codes



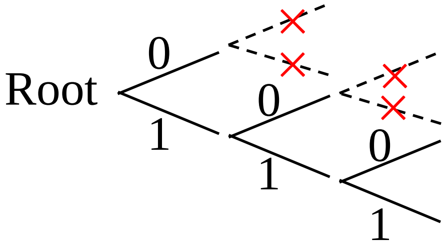
- At level $l_{\max}$ of the tree (source codeword length is $l_{\max}$), there are at most $D^{l_{\max}}$ codewords. Similarly, at level l_i of the tree, there are at most D^{l_i} codewords. A codeword at level l_i has $D^{l_{\max}-l_i}$ descendants at level $l_{\max}$.

- The descendent sets of all codewords should be <u>disjoint</u>. Consider all codewords, this property implies

$$\sum_i D^{l_{\max}-l_i} \le D^{l_{\max}} \longrightarrow \sum_i D^{-l_i} \le 1.$$

§ 3.2 Optimal Source Codes

- The tree represents an instantaneous source code.



- The expected length of this tree is

$$\mathbb{E}[l] = \sum_i l_i p_i$$

l_i: length of a source codeword for symbol x_i

p_i: probability of symbol x_i

- Expected length of the tree is the expected length of the source code.

§ 3.2 Optimal Source Codes

- Finding the smallest expected length L becomes

$$\boxed{\begin{aligned}\text{minimize:}\quad & L=\sum_i p_i l_i\\ \text{s.t.}\quad & \sum_i D^{-l_i}\le 1.\end{aligned}}$$

- The constrained minimization problem can be interpreted through the Lagrange multipliers as:

$$\boxed{\text{minimize:}\quad J=\sum_i p_i l_i+\lambda\left(\sum_i D^{-l_i}\right)}$$

- Calculus: $\frac{\partial J}{\partial l_i}=p_i-\lambda D^{-l_i}\log_e D$. To enable $\frac{\partial J}{\partial l_i}=0$, we need $D^{-l_i}=\frac{p_i}{\lambda\log_e D}$.

- To satisfy the Kraft Inequality, we have $\lambda=\frac{1}{\log_e D}$. Hence, $p_i=D^{-l_i}$.

- To minimized L, we need $l_i^*=\log_D p_i^{-1}$.

- With $l_i^*=\log_D p_i^{-1}$, we have

$$L=\sum_i p_i l_i^*=\sum_i p_i\log_D p_i^{-1}=\underline{H_D(X)}$$

Entropy of the source symbols

§ 3.2 Optimal Source Codes

Theorem 3.3 (Lower Bound of the Expected Length) The expected length L of an instantaneous D-ary code for a random variable X is lower bounded by

$$L \geq H_D(X).$$

Proof:

$$\begin{aligned} L - H_D(X) &= \sum_i l_i p_i + \sum_i p_i \log_D p_i \\ &= -\sum_i p_i \log_D D^{-l_i} + \sum_i p_i \log_D p_i \\ &= \sum_i p_i \log_D \frac{p_i}{D^{-l_i}}. \end{aligned}$$

Let $p_i' = \frac{D^{-l_i}}{\sum_i D^{-l_i}} = \frac{D^{-l_i}}{V}$,

§ 3.2 Optimal Source Codes

$$
\begin{aligned}
L - H_D(X) &= \sum_i p_i \log_D \frac{p_i}{p_i' \cdot V} \\
&= \sum_i p_i \log_D \frac{p_i}{p_i'} - \sum_i p_i \log_D V \\
&= D(p_i || p_i') + \sum_i p_i \log_D \frac{1}{V} \\
&\geq 0.
\end{aligned}
$$

Note, when $p_i' = p_i, \forall i$, $D(p_i || p_i') = 0, V = 1$ and $\sum_i p_i \log_D \frac{1}{V} = 0$.

Remark: since l_i can be only be an integer,

$$
\begin{aligned}
L &= H_D(X),\ \text{if } l_i = -\log_D p_i. \\
L &> H_D(X),\ \text{if } l_i = \lceil -\log_D p_i \rceil.
\end{aligned}
$$

§ 3.2 Optimal Source Codes

Corollary 3.4 (Upper Bound of the Expected Length) The expected length L of an instantaneous *D*-ary code for a random variable X is upper bounded by

$$L < H_D(X) + 1.$$

Proof: Since $-\log_D p_i \le l_i < -\log_D p_i + 1$.

By multiplying p_i to the above inequality and performing summation over i as

$$\sum_{i} -p_i \log p_i \le \sum\nolimits_i p_i l_i < \sum\nolimits_i -p_i \log p_i + \sum\nolimits_i p_i$$

$$H_D(X) \le L < H_D(X) + 1.$$

§ 3.3 Shannon-Fano Code

- Given a source that contains symbols $x_1, x_2, \dots, x_U$ with probabilities of $p_1, p_2, \dots, p_U$, respectively.
- Determine the source codeword length for symbol x_i as

$$l_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil \text{ bits.}$$

- Further determine $l_{\max} = \max\{l_i, \forall i\}$.
- **Shannon-Fano Code Construction**:

Step 1: Construct a binary tree of depth $l_{\max}$.

Step 2: Choose a node of depth l_i and delete its following paths and nodes. The path from root to the node represents the source codeword for source symbol x_i.

§ 3.3 Shannon-Fano Code

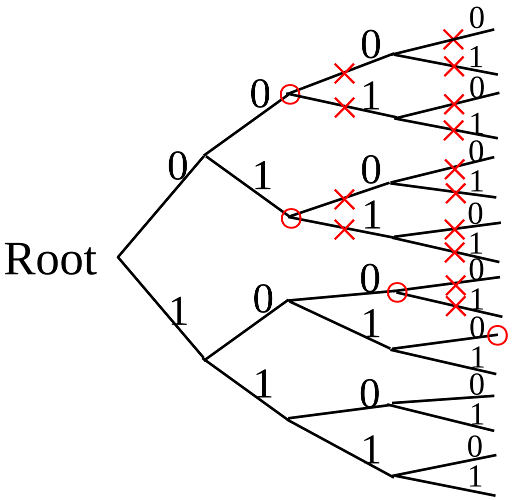
- ***Example 3.3*** Given a source with symbols x_1, x_2, x_3, x_4, they occur with a probability of $p_1 = 0.4$, $p_2 = 0.3$, $p_3 = 0.2$, $p_4 = 0.1$, respectively. Construct its Shannon-Fano code.

We can determine

$$l_1 = \left\lceil \log_2 \frac{1}{p_1} \right\rceil = 2, l_2 = \left\lceil \log_2 \frac{1}{p_2} \right\rceil = 2, l_3 = \left\lceil \log_2 \frac{1}{p_3} \right\rceil = 3, l_4 = \left\lceil \log_2 \frac{1}{p_4} \right\rceil = 4,$$

and $l_{\max} = 4$.

Construct a binary tree of depth 4.



The source codewords are

x_1: 0 0

x_2: 0 1

x_3: 1 0 0

x_4: 1 0 1 0.

Note: $L = 2.4$ bits/sym., $H(X) = 1.85$ bits/sym.,

and $H(X) < L < H(X) + 1$.

§ 3.4 Huffman Code

- Given a source that contains symbols $x_1, x_2, \ldots, x_U$ with probabilities of $p_1, p_2, \ldots, p_U$, respectively.

- **Huffman Code Construction**:

  Step 1: Merge the 2 smallest symbol probabilities;

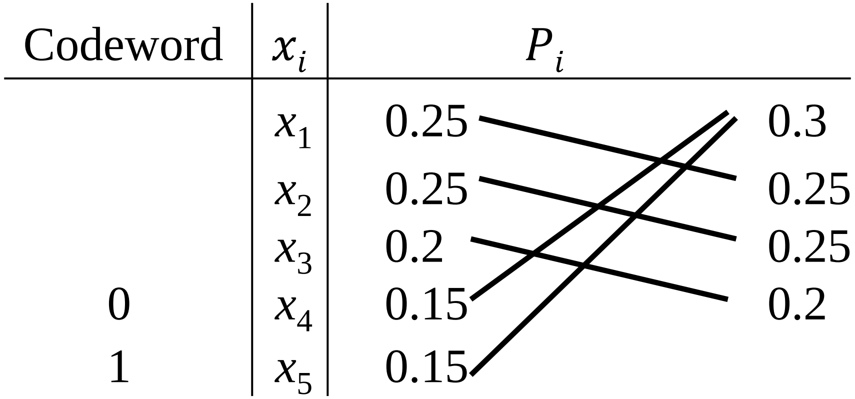
  Step 2: Assign the 2 corresponding symbols with 0 and 1, then go back to **Step 1**;

  Repeat the above process until two probabilities are merged into a probability of 1.

- Huffman code is the shortest prefix code, i.e., an optimal code.

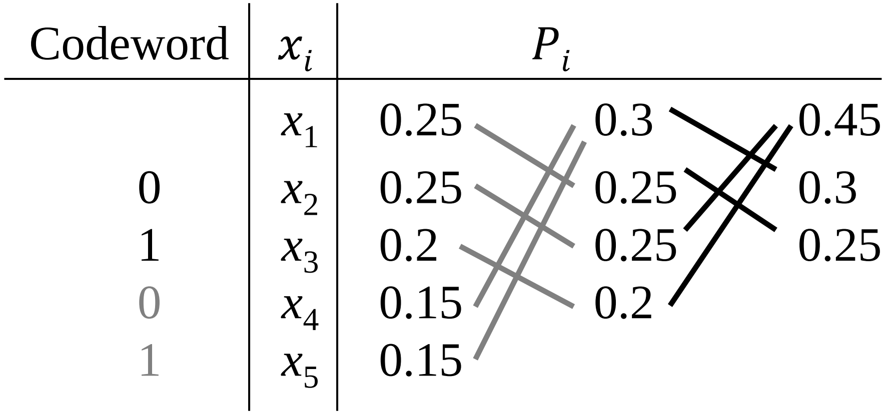
§ 3.4 Huffman Code

Example 3.4 Given a source with symbols x_1, x_2, x_3, x_4, x_5. They occur with probabilities of $P_1 = 0.25,\ P_2 = 0.25, P_3 = 0.2,\ P_4 = 0.15,\ P_5 = 0.15$, respectively. Construct its Huffman code.

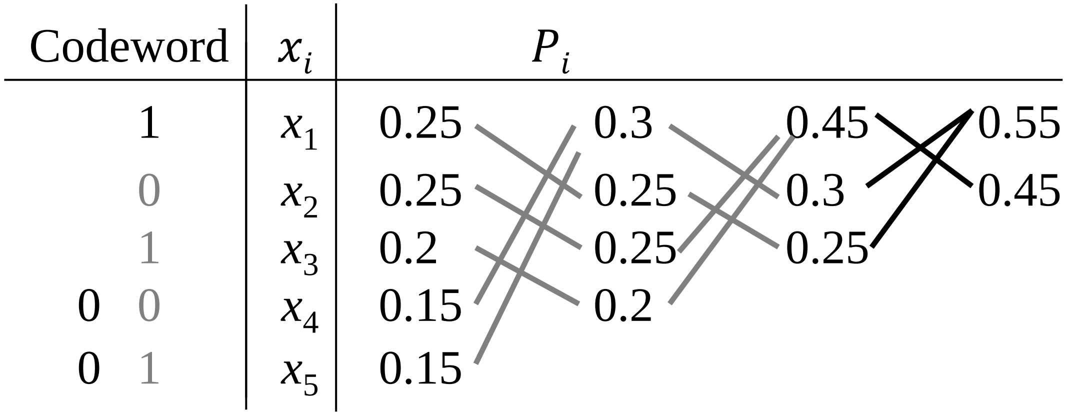


Codeword	x_i	P_i	
	x_1	0.25	0.3
	x_2	0.25	0.25
	x_3	0.2	0.25
0	x_4	0.15	0.2
1	x_5	0.15	

§ 3.4 Huffman Code

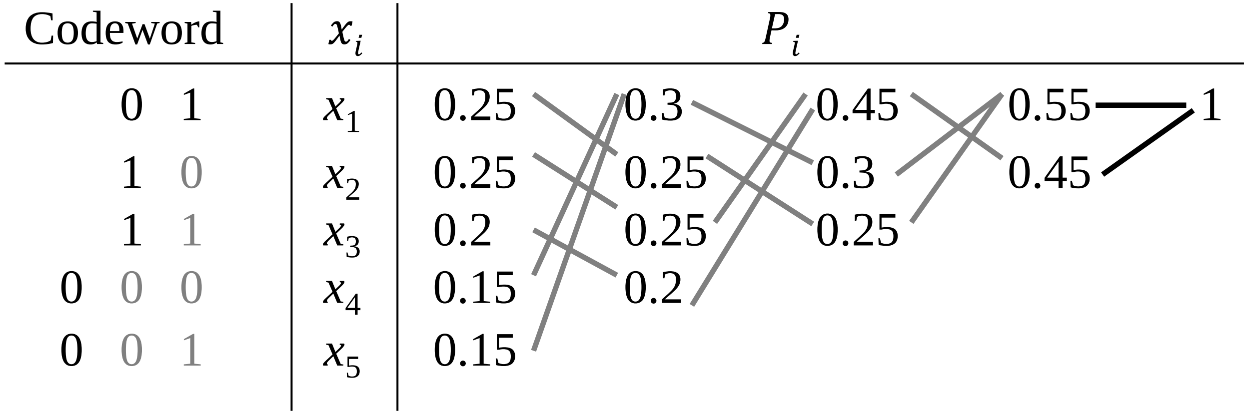


Codeword	x_i	P_i		
	x_1	0.25	0.3	0.45
0	x_2	0.25	0.25	0.3
1	x_3	0.2	0.25	0.25
0	x_4	0.15	0.2	
1	x_5	0.15		



Codeword		x_i	P_i			
	1	x_1	0.25	0.3	0.45	0.55
	0	x_2	0.25	0.25	0.3	0.45
	1	x_3	0.2	0.25	0.25	
0	0	x_4	0.15	0.2		
0	1	x_5	0.15			

§ 3.4 Huffman Code



Codeword	x_i	P_i				
0 1	x_1	0.25	0.3	0.45	0.55	1
1 0	x_2	0.25	0.25	0.3	0.45	
1 1	x_3	0.2	0.25	0.25		
0 0 0	x_4	0.15	0.2			
0 0 1	x_5	0.15				

Validations:

$l_1 = 2, l_2 = 2, l_3 = 2, l_4 = 3, l_5 = 3$

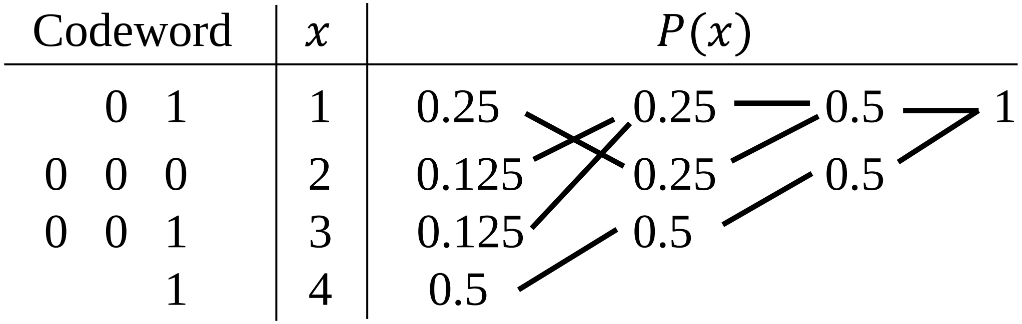
$L = \sum_i l_i \cdot P_i = 2.3$ bits/symbol

$H_2(X) = \sum_i P_i \log_2 P_i^{-1} = 2.3$ bits/sym.

Q: Try to construct a Shannon-Fano code and see if it is also optimal.

§ 3.4 Huffman Code

So now, let us look back at the problem proposed at the beginning.
How to represent the source vector {1 2 4 4 3 1 4 4} ?



Codeword	x	$P(x)$			
0 1	1	0.25	0.25	0.5	1
0 0 0	2	0.125	0.25	0.5	
0 0 1	3	0.125	0.5		
1	4	0.5			

It should be represented as {0 1 0 0 0 1 1 0 0 1 0 1 1 1 } and $L = 1.75$ bits/symbol.

Q: How if the source vector becomes {1 2 4 3 4 4 2 1}?

Remark: The source coding depends on the source vector.

§ 3.4 Huffman Code

- Huffman code can also be defined as a *D*-ary code.

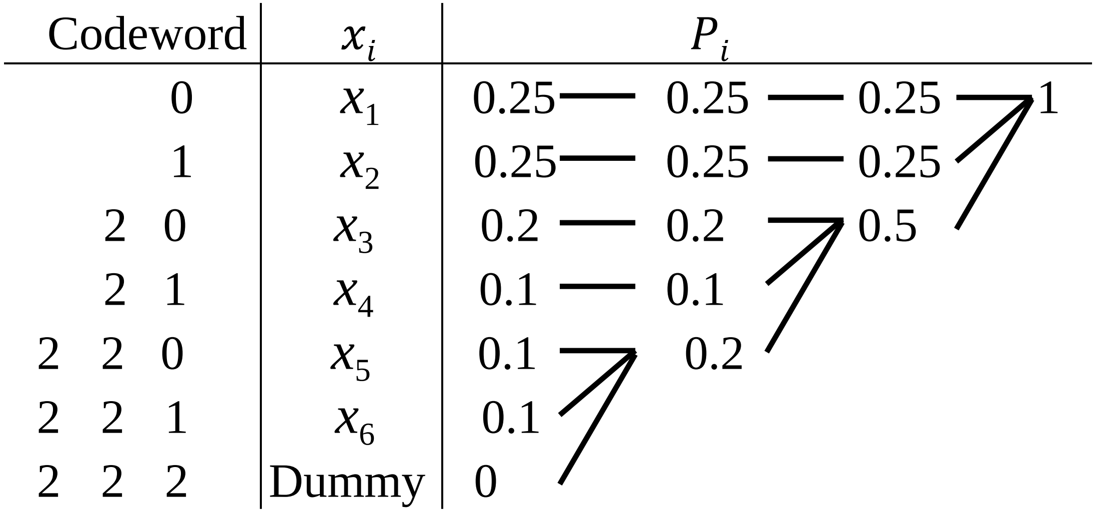
- A *D*-ary Huffman code can be similarly constructed following the binary construction.

Step 1: Merge the *D* smallest symbol probabilities;

Step 2: Assign the corresponding symbols with 0, 1, ..., $D - 1$, then go back to **Step 1**; Repeat the above process until *D* probabilities are merged into a probability of 1.

§ 3.4 Huffman Code

Example 3.5 Consider a source with symbols x_1, x_2, x_3, x_4, x_5, x_6. They occur with probabilities of $P_1 = 0.25$, $P_2 = 0.25$, $P_3 = 0.2$, $P_4 = 0.1$, $P_5 = 0.1$, $P_6 = 0.1$, respectively. Construct a ternary ({0, 1, 2}) Huffman code.



Codeword	x_i	P_i			
0	x_1	0.25	0.25	0.25	1
1	x_2	0.25	0.25	0.25	
2 0	x_3	0.2	0.2	0.5	
2 1	x_4	0.1	0.1		
2 2 0	x_5	0.1	0.2		
2 2 1	x_6	0.1			
2 2 2	Dummy	0			

Note: A dummy symbol is created such that 3 probabilities can merge into a probability of 1 in the end.

§ 3.4 Huffman Code

Properties on an optimal *D*-ary source code (Huffman code)

(1) If $p_j > p_k$, then $l_j \leq l_k$;

(2) The D longest codewords have the same length;

(3) The D longest codewords differ only at the last symbol and correspond to the D least likely source symbols.

Theorem 3.5 (Optimal Source Code) A source code (C^*) is optimal if giving any other source code C', we have $L(C^*) \leq L(C')$.

Note: Huffman codes are optimal.

§ 3.5* Source Coding Schemes : Arithmetic Code

- Arithmetic code is a lossless compression method that encodes the message into a fractional number between 0 and 1.
- Higher precision in fractional number requires more bits, enabling to encode longer message.
- The interval [0,1) is partitioned based on the source distribution and the symbol sequence of length n.

§ 3.5* Source Coding Schemes : Arithmetic Code

- Given a source whose symbols are chosen from the alphabet $\{x_1, x_2, x_3, \dots, x_U\}$ with the alphabet symbol probabilities of $p_1, p_2, p_3, \dots, p_U$, respectively.
- Given cumulative distribution function $F(x)$.
- Let $F(x_0) = 0$, x_0 is an imaginary symbol.

$$F(x_1) = p_1$$

$$F(x_2) = p_1 + p_2$$

$$\vdots$$

$$F(x_U) = p_1 + p_1 + \cdots + p_U = 1$$

Note : $F(x_i) - F(x_{i-1}) = p_i$.

§ 3.5* Source Coding Schemes : Arithmetic Code

- Encoding a source symbol sequence $s_1, s_2, \cdots, s_n$

  Initialization

  $A^{(0)} = 0$ $B^{(0)} = 1$

  Encode symbol x_i at time instant t, i.e., $s_t = x_i$

  $$A^{(t)} = A^{(t-1)} + (B^{(t-1)} - A^{(t-1)})\, F(x_{i-1})$$

  $$B^{(t)} = A^{(t-1)} + (B^{(t-1)} - A^{(t-1)})\, F(x_i)$$

Note : $B^{(t)} - A^{(t)} = \left(B^{(t-1)} - A^{(t-1)}\right)\left(F(x_i) - F(x_{i-1})\right)$

$$= \left(B^{(t-1)} - A^{(t-1)}\right)p_i$$

§ 3.5* Source Coding Schemes : Arithmetic Code

Example 3.6 Given a source with symbols x_1, x_2, x_3 and their probabilities $p_1 = 0.6, p_2 = 0.3, p_3 = 0.1$.

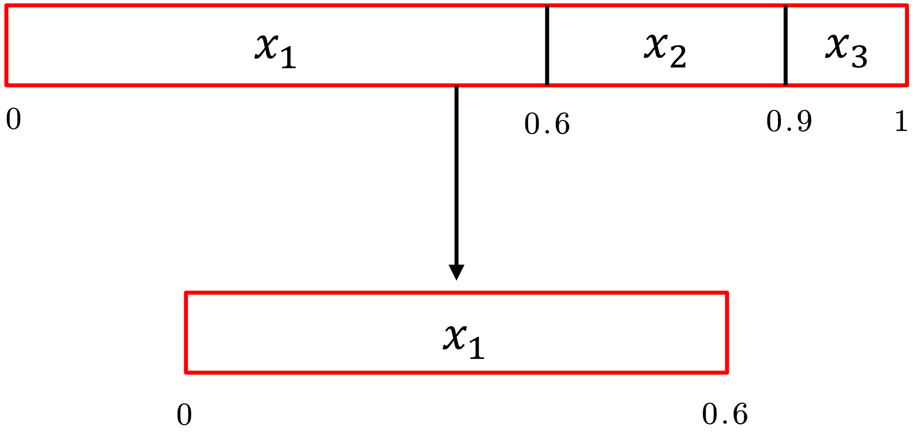
Given the sequence $s_1 = x_1,\ s_2 = x_3, s_3 = x_2,\ s_4 = x_1$.

Determine the value of cumulative distribution function $F(x)$:

$$F(x_1) = 0.6, F(x_2) = 0.9, F(x_3) = 1.0, F(x_0) = 0$$

For $t = 1$, encode x_1:

$$A^{(1)} = A^{(0)} + \left(B^{(0)} - A^{(0)}\right) \cdot F(x_0) = 0,$$
$$B^{(1)} = A^{(0)} + \left(B^{(0)} - A^{(0)}\right) \cdot F(x_1) = 0.6$$



§ 3.5* Source Coding Schemes : Arithmetic Code

For $k = 2$, encode x_3:

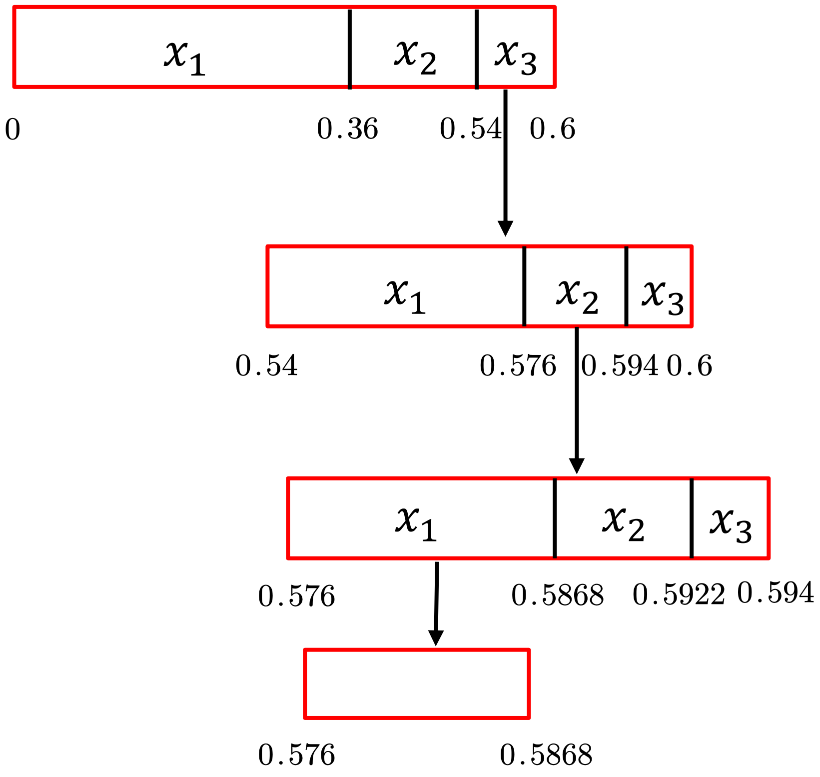
$$A^{(2)} = A^{(1)} + \left(B^{(1)} - A^{(1)}\right) \cdot F(x_2) = 0.54\ ,$$
$$B^{(2)} = A^{(1)} + \left(B^{(1)} - A^{(1)}\right) \cdot F(x_3) = 0.6$$

For $k = 3$, encode x_2:

$$A^{(3)} = A^{(2)} + \left(B^{(2)} - A^{(2)}\right) \cdot F(x_1) = 0.576\ ,$$
$$B^{(3)} = A^{(2)} + \left(B^{(2)} - A^{(2)}\right) \cdot F(x_2) = 0.594$$

For $k = 4$, encode x_1:

$$A^{(4)} = A^{(3)} + \left(B^{(3)} - A^{(3)}\right) \cdot F(x_0) = 0.576\ ,$$
$$B^{(4)} = A^{(3)} + \left(B^{(3)} - A^{(3)}\right) \cdot F(x_1) = 0.5868$$



Finally, we get the internal $[0.576, 0.5868)$.

§ 3.5* Source Coding Schemes : Arithmetic Code

For $[0.576, 0.5868)$, determine the fractional number c that can be represented by a finite binary expression.

Initial $a = 0, b = 1, c = \frac{a+b}{2} = 0.5$.

```
While $c \notin [0.576, 0.5868)$   (Binary search)
    if $c < 0.576$
      $a = c;$
    else if $c \geq 0.5868$
      $b = c;$
    $c = \frac{a+b}{2};$
End
Output : $c = 0.578125$
```

§ 3.5* Source Coding Schemes : Arithmetic Code

Convert the fractional number 0.578125 to its binary representation.

$$\begin{aligned} 0.578125 \times 2 &= 1.15625 \\ 0.15625 \times 2 &= 0.3125 \\ 0.3125 \times 2 &= 0.625 \\ 0.625 \times 2 &= 1.25 \\ 0.25 \times 2 &= 0.5 \\ 0.5 \times 2 &= 1 \end{aligned}$$

$$0.100101_2 = 2^{-1} + 2^{-4} + 2^{-6} = 0.578125$$

The sequence $x_1 x_3 x_2 x_1$ is encoded to 0.100101_2.

§ 3.5* Source Coding Schemes : Arithmetic Code

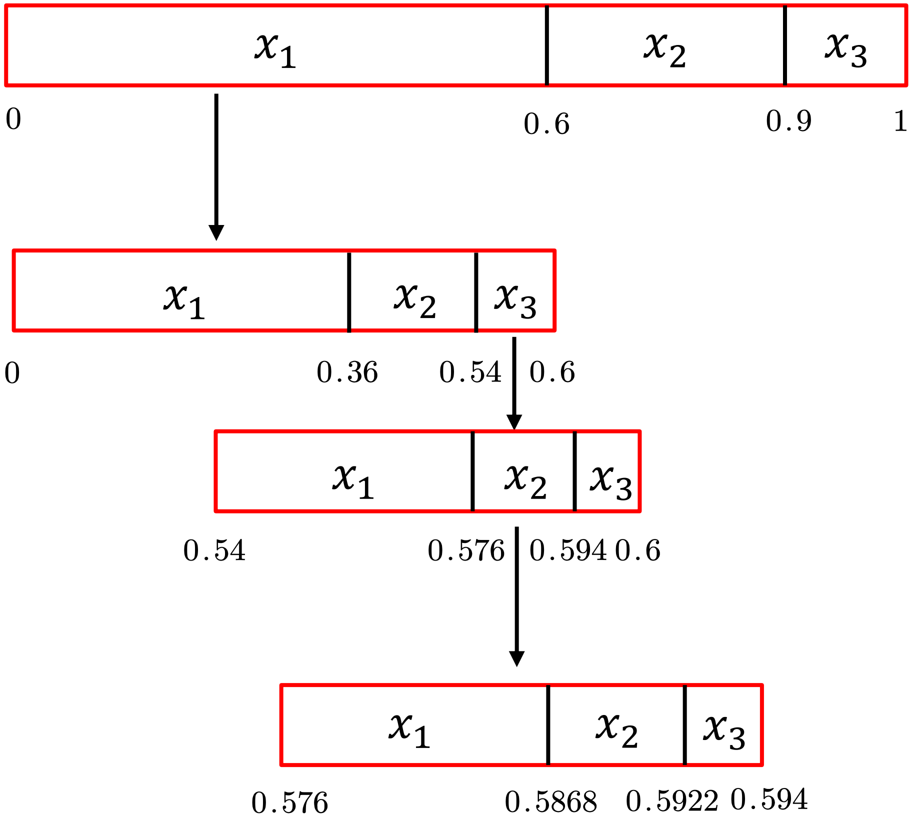
Decoding Arithmetic code 0.100101_2 (0.578125)

$0 < 0.578125 < 0.6$, output: x_1

$0.54 < 0.578125 < 0.6$, output: x_3

$0.576 < 0.578125 < 0.594$, output: x_2

$0.576 < 0.578125 < 0.5868$, output: x_1



Finally, the bits 0.100101_2 is decoded to the sequence $x_1x_3x_2x_1$.

§ 3.5* Source Coding Schemes : Arithmetic Code

Theorem 3.6 Entropy Coding An entropy coding is any lossless source coding with an expected length greater than or equal to the source entropy.

- Note: Arithmetic code is an entropy coding.

Proof:

- Source : $X = \{x_1, x_2, \dots, x_U\}$
- Symbol sequence : $s_1, s_2, \cdots, s_n$
- Average length of per symbol: l
- Length of the final coding interval: $\beta = \prod_{k=1}^{n} p(s_k)$
- The minimum number of bits to represent the internal: $\lceil -\log_2 \beta \rceil = -\log_2 \beta + \sigma$

§ 3.5* Source Coding Schemes : Arithmetic Code

$$l \leq \frac{\lceil -\log_2 \beta \rceil}{n} = \frac{-\log_2 \beta + \sigma}{n}$$

$$l \leq \frac{-\sum_{k=1}^{n} \log_2 p(s_k) + \sigma}{n}$$

$$E(l) \leq \frac{\sum_{k=1}^{n} E\left(-\log_2 p(s_k)\right) + \sigma}{n}$$

$$= \underline{H^*(x)} + \frac{\sigma}{n}$$

$$E(l) \leq H(X) + \frac{\sigma}{n}$$

Empirical entropy

Note : If the distribution of X is known, $H^*(X) = H(X)$.

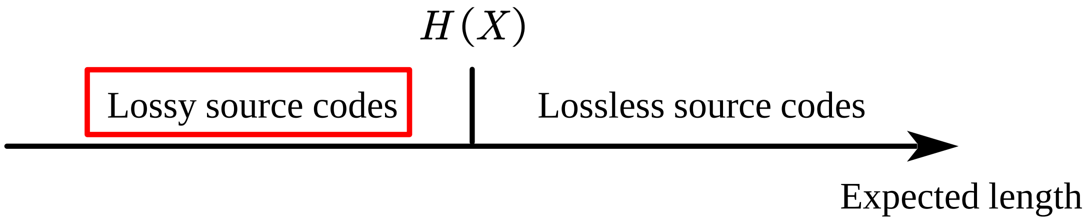
- When $n = \infty$, the expected code length of Arithmetic code can reach the entropy of the source.

§ 3.6* Rate-Distortion

- Consider a source coding with its rate (expected length) less than the source entropy.

- Under such a situation, source must be compressed with distortion.

- What is the best possible trade-off between rate and distortion?

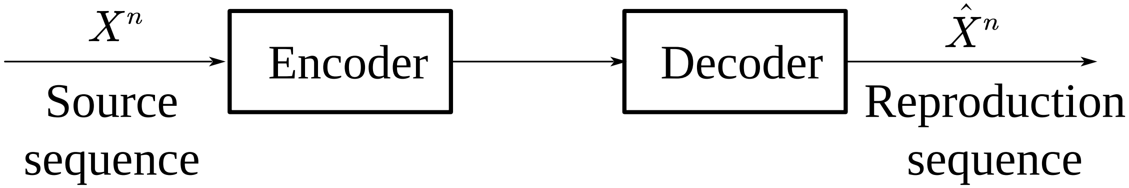


§ 3.6* Rate-Distortion

- Source alphabet: $\mathcal{X}$ and reproduction alphabet: $\widehat{\mathcal{X}}$.
- Single-letter distortion measure: $d: \mathcal{X} \times \widehat{\mathcal{X}} \to \mathcal{R}^{+}$.
- The value $d(x, \hat{x})$ denotes the distortion incurred when a source symbol $x \in \mathcal{X}$ is reproduced as $\hat{x} \in \widehat{\mathcal{X}}$.
- Hamming distortion: $d(x, \hat{x}) = \begin{cases} 0, & \text{if } x = \hat{x}; \\ 1, & \text{if } x \neq \hat{x}. \end{cases}$
- The distortion between sequences x^n and $\hat{x}^n$: $d(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i)$.

§ 3.6* Rate-Distortion

- Let $\{X_k, 1 \leq k \leq n\}$ be an i.i.d. information source with random variable $X \sim p(x), x \in \mathcal{X}$.



A $(2^{nR}, n)$ rate-distortion code :

Length of source sequence : n.

Length of this rate-distortion code : 2^{nR}.

Rate of this rate-distortion code : $R = \frac{\log_2 2^{nR}}{n}$.

§ 3.6* Rate-Distortion

A $(2^{nR}, n)$ rate-distortion code is defined by two functions :

Encoding function : $f_n\colon \mathcal{X}^n \to \{1,2,\cdots,2^{nR}\}$.

Decoding function : $g_n\colon \{1,2,\cdots,2^{nR}\} \to \widehat{\mathcal{X}}^n$.

Distortion of this rate-distortion code : $D = \mathbb{E}d(X^n, g_n(f_n(X^n)))$

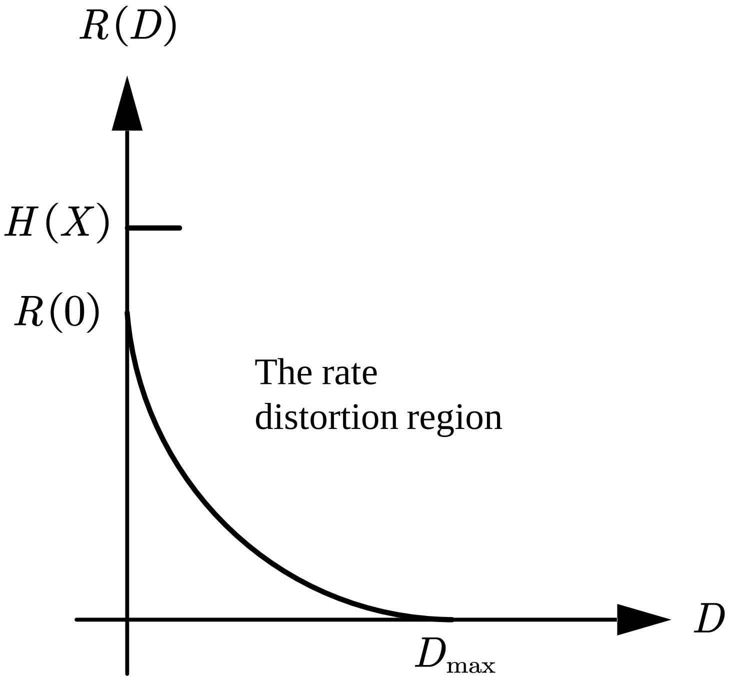
$$= \sum_{x^n} p(x^n) d\left(x^n, g_n(f_n(x^n))\right).$$

Definition 3.1: A rate-distortion pair (R, D) is said to be achievable if there exists a sequence of $(2^{nR}, n)$ rate-distortion codes (f_n, g_n) with

$$\lim_{n\to\infty} \mathbb{E}d\left(X^n, g_n(f_n(X^n))\right) \le D.$$

§ 3.6* Rate-Distortion

- **Definition 3.2 :** Rate-distortion function $R(D)$ is the minimum of all rates R for a given distortion D such that (R, D) is achievable.



§ 3.6* Rate-Distortion

Properties of the rate-distortion function $R(D)$:

(1). $R(D)$ is non-increasing in D.

Let $D' \geq D$. $(R(D), D)$ achievable $\Rightarrow$ $(R(D), D')$ achievable.

(2). $R(D)$ is convex.

Follows from the convexity of the rate-distortion region.

(3). $R(D) = 0$ for $D \geq D_{\max}$.

$(0, D_{\max})$ is achievable $\Rightarrow$ $R(D) = 0$ for $D \geq D_{\max}$.

(4). $R(D) \leq H(X)$.

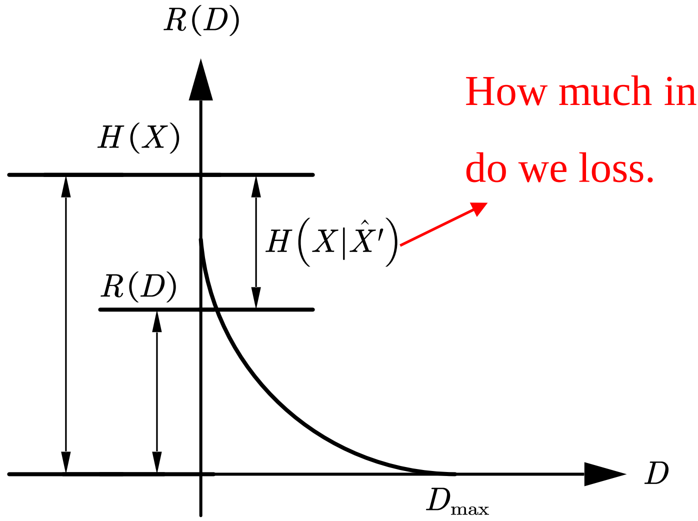
The rate-distortion code is a lossy encoding.

§ 3.6* Rate-Distortion

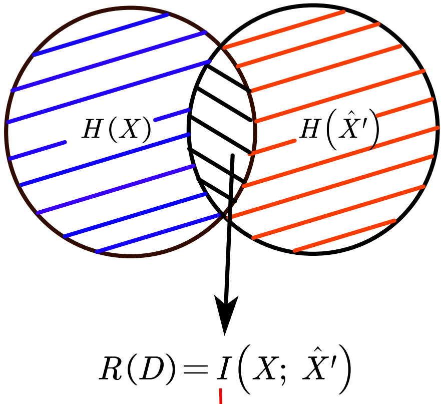
Definition 3.3 : Information rate-distortion function : $R^{(I)}(D) = \min_{\hat{X}:\mathbb{E}d(X,\hat{X})\le D} I(X;\hat{X})$

Theorem 3.7 Rate-distortion theorem $R^{(I)}(D) = R(D)$.

Assume that $\min_{\hat{X}:\mathbb{E}d(X,\hat{X})\le D} I(X;\hat{X}) = I(X;\hat{X}')$



How much information do we loss.



How much information do we know about X through source code of rate $R(D)$.

$R(D)$ is the minimum achievable rate for lossless compression of the residual information $I(X;\hat{X}')$.

§ 3.6* Rate-Distortion

Example 3.7 Given a binary source that satisfies $\Pr\{X=0\}=1-\gamma$ and $\Pr\{X=1\}=\gamma$ and distortion D. Using Hamming distortion as distortion measure, let we assume that $\mathbb{E}d(X,\hat{X})=\Pr\{X\neq\hat{X}\}\leq D$. Determine the $R(D)$.

Without loss of generality, we may assume that $\gamma\leq\frac{1}{2}$.

$$\begin{aligned}
I(X;\hat{X}) &= H(X)-H(X|\hat{X})\\
&= H_b(\gamma)-H(X\oplus\hat{X}|\hat{X})\\
&\geq H_b(\gamma)-H(X\oplus\hat{X})\\
&= H_b(\gamma)-H_b\left(\Pr\{X\neq\hat{X}\}\right)
\end{aligned}$$

Note : $H_b(\gamma)=-\gamma\log_2\gamma-(1-\gamma)\log_2(1-\gamma)$

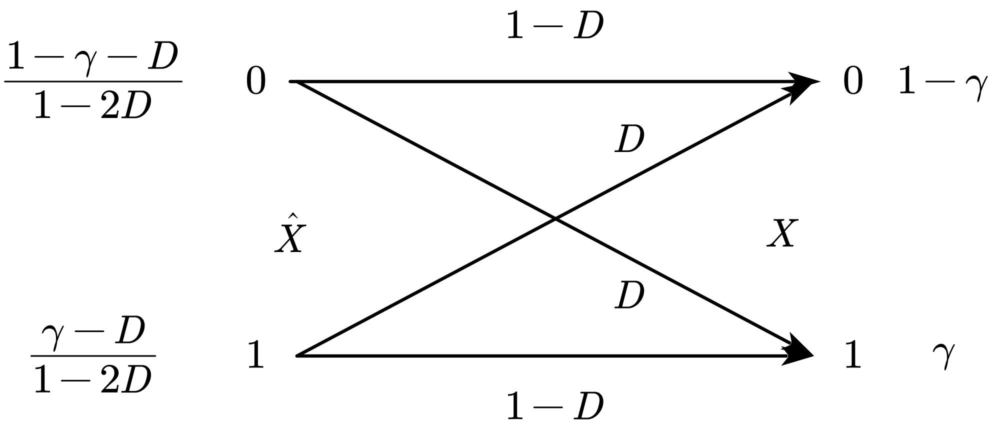
$\Pr\{\mathrm{X}\neq\hat{X}\}\leq D$ and $H_b(\gamma)$ increases with γ for $\gamma\leq\frac{1}{2}$

$$I(X;\hat{X})\geq H_b(\gamma)-H_b(D)$$

§ 3.6* Rate-Distortion

$$R^{(I)}(D) = R(D) = \min_{\hat{X}:\mathbb{E}d(X,\hat{X})\leq D} I\left(X;\hat{X}\right) \geq H_b(\gamma) - H_b(D).$$

When $\gamma \geq D$



For this BSC, $I\left(X;\hat{X}\right) = H(X) - H\left(X\middle|\hat{X}\right)$

$$= H_b(\gamma) - (-D\log_2 D - (1-D)\log_2(1-D))$$

$$= H_b(\gamma) - H_b(D).$$

§ 3.6* Rate-Distortion

Information rate distortion function:

$$R^{(I)}(D) = R(D) = \min_{\hat{X}:\mathbb{E}d(X,\hat{X})\leq D} I\big(X;\hat{X}\big)$$

$$= H_b(\gamma) - H_b(D), \text{ where } \gamma < \frac{1}{2} \text{ and } \gamma \geq D.$$

$$\downarrow$$

$$R(D) = \begin{cases} H_b(\gamma) - H_b(D), & \text{if } 0 \leq D \leq \min(\gamma, 1-\gamma)\,; \\ 0, & \text{if } D \geq \min(\gamma, 1-\gamma)\,. \end{cases}$$

References:

[1] Elements of Information Theory, by T. Cover and J. Thomas.

[2] Scriptum for the lectures, Applied Information Theory, by M. Bossert.